4 A glimpse of the nonlinear world. Hopf's equation

Recall that I obtained the linear transport equation $u_t+cu_x=0$ as the consequence of the conservation law $u_t+q_x=0$, which connects the quantity u and its flux q, and the assumption that $q(u)=cu$. If I make another assumption, namely $q(u)=u^2/2$, I will end up with the so-called *Hopf's equation* (there are a lot of other names for this equation, i.e., the *inviscid Burgers' equation* or *nonlinear transport equation*)

$$u_t+uu_x=0, \tag{4.1}$$

for which I will need the initial condition

$$u(0,x)=g(x) \tag{4.2}$$

for some given smooth function g.

Equation (4.1) is an example of a *quasi-linear* equation. All quasi-linear equations can be analyzed using a slight modification of the characteristic method, but I will only look at solutions of the Hopf equation.

To solve (4.1)-(4.2) I will use an analogy with the transport equation and assume that $t\mapsto x(t)$ is my characteristic (which I do not know yet, and which in general should also depend on parameter ξ, which I omitted for simplicity), which solves

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)=u(t,x(t)).$$

Here is the first problem: I do not know u yet, so I cannot solve this equation. Note, however, that if $h(t)=u(t,x(t))$ is my solution along the (unknown) characteristic, then

$$h'(t)=u_t(t,x(t))+x'(t)u_x(t,x(t))=0$$

according to the equation, and therefore I know that my solution along the characteristic is constant! Since it is constant and I know it at one point along this characteristic, namely, $u(0,x(0,\xi))=g(\xi)$, where, as before, I use the fact that $x(0,\xi)=\xi$ is my parametrization of the initial condition, hence

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)=g(\xi) \implies x(t)=g(\xi)t+\xi,$$

which means that my characteristics are the straight lines, but with the slope that depends on the initial condition.

To actually get a formula for my solution (which is not explicit though, see below), I start with

$$u(t,x(t,\xi))=g(\xi)=g(x(t,\xi)-g(\xi)t)=g\big(x(t,\xi)-u(t,x(t,\xi))t\big),$$

which gives, after dropping the explicit dependence x on t and ξ,

$$u(t,x)=g\big(x-u(t,x)t\big), \tag{4.3}$$

which gives me the solution to my problem in an implicit form.

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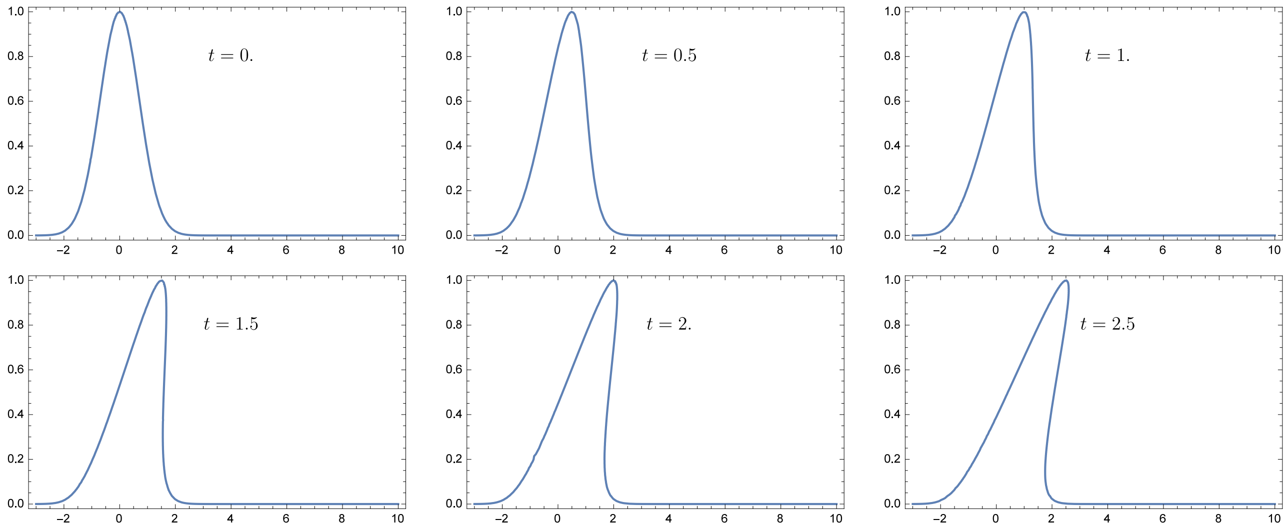

Figure 1: Nonlinear wave in Hopf's equation.

Exercise 1. Check directly that $u(t,x) = g\big(x - u(t,x)t\big)$ solves problem (4.1)-(4.2).

To see the possible consequence of (4.3) consider the following example.

Example 4.1. Let

$$g(x) = e^{-x^2}.$$

Then, using (4.3), I have

$$u = e^{-(x-ut)^2},$$

which is impossible to resolve for u, but I can sketch my solution numerically for several time moments, see Fig. 1. We can see that for our nonlinear problem the velocity of propagation depends on the spatial coordinate and the initial density. Basically the equation itself says that the points with higher concentration move faster, with time overcoming the points with lower concentration.

To understand what is happening here it is worth drawing the characteristics (see Fig. 2). Since the slope of the characteristics depends on the initial concentrations, in some cases, as in my example, the characteristics can intersect. Since my solution must be constant along the characteristics it means that at the point where the characteristics intersect I must have more than one value of u, which is what exactly happens in Fig. 1.

To find the first time moment when the characteristics intersect, I can look at the point when the derivative of u with respect to x becomes infinite. Using the implicit differentiation, I find from (4.3)

$$\frac{\partial u}{\partial x} = g'(x-ut)\left(1 - \frac{\partial u}{\partial x}t\right) \implies \frac{\partial u}{\partial x} = \frac{g'(x-ut)}{1+g'(x-ut)t}\,.$$

Recalling that $x - ut = \xi$ I finally get the condition of the first time when my solution becomes multivalued (the characteristics intersect) as

$$t = \min_{\xi}\left\{-\frac{1}{g'(\xi)}\right\}.$$

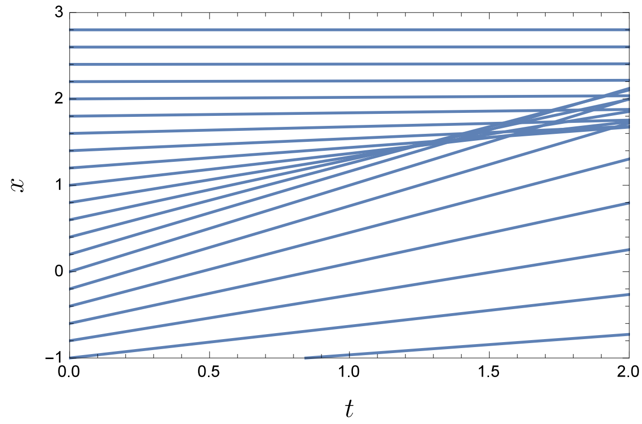

Figure 2: Characteristics of Hopf's equation with the initial condition $g(x) = e^{-x^2}$.

For my example, I get

$$t = \frac{\sqrt{2e}}{2} \approx 1.166\,.$$

Exercise 2. Confirm this value.

Note that if $g'(\xi) > 0$ then my solution is well defined for all $t > 0$.

Equations of the form (4.1) appear in various situations, in particular in gas or hydro- dynamics, or in the modeling of the car traffic. From a physical point of view to have several values of the solution at the point x is meaningless and hence must be avoided. The exact details how it is done can be found in various (usually graduate level) textbooks (see, e.g., Whitham, G.B. Linear and nonlinear waves. John Wiley & Sons, 2011).

As a take home message from this lecture I would like all the students to remember that the main distinction between linear (solutions to the transport equation) and nonlinear (solutions to the Hopf equation) waves is that for the linear wave all the points move with the same velocity whereas for the nonlinear waves the velocity may depend on the concentration. If someone wants to have a more colorful picture of this phenomenon remember the waves at the ocean, when eventually the wave collapses: this is a nice example of a nonlinear wave.

4.1 Test yourself

4.1. What is the condition on the initial condition g for Hopf's equation that guarantees that the characteristics do not intersect for $t > 0$?

4.2. Consider Hopf's equation together with the initial condition

$$g(x) = \alpha x + \beta,$$

where α and β are given numbers. Find the solution in this case. Discuss the behavior of solutions for the cases $\alpha > 0$ and $\alpha < 0$ for $t > 0$. For which α the characteristics intersect? Find the critical time value when the characteristics intersect.

Figure 3: An example of a nonlinear wave in nature.

4.2 Solutions to the exercises

Exercise 1. This is basically an exercise in implicit differentiation. From $u = g(x - ut)$ and recalling that u is a function of t and x, I have, taking derivative with respect to t from both the left and right hand sides

$$u_t = g'(x - ut)(-u_t\,t - u) \implies u_t = -\frac{ug'(x - ut)}{1 + g'(x - ut)t}\,.$$

Similarly, taking the derivative with respect to x:

$$u_x = g'(x - ut)(1 - u_x\,t) \implies u_x = \frac{g'(x - ut)}{1 + g'(x - ut)t}\,.$$

Hence

$$u_t + uu_x = 0$$

as required. ∎

Exercise 2. Here $g(\xi) = e^{-\xi^2}$ and we need to minimize the function

$$h(\xi) = -\frac{1}{g'(\xi)} = \frac{e^{\xi^2}}{2\xi}\,.$$

It would help to make a sketch of the graph of this function, but I will proceed directly, by finding that the equation $h'(\xi) = 0$ has two roots $\xi_\pm = \pm\frac{1}{\sqrt{2}}$, and it is the positive root that is of interest. For this root $h(\xi_+) = \frac{\sqrt{2e}}{2}$ as indicated above. ∎